Appendices for:

Search for Financial Returns and Social Security Privatization

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A Proofs

Proofs for section 2

Lemma 1. *Under assumption 1, if $F(r)$ is continuous in r, then*

$$p(a, \epsilon; r) \equiv \mu^{-1} \sum_{j=0}^{\infty} j q\left(j; a, \epsilon\right) \left[F(r)\right]^{j-1}$$

is continuous in r for any (a, ϵ).

Proof. Consider two points r^1 and r^2, and let $F(r^1) \leq F(r^2)$ without loss of generality. Let $z = F(r^2) - F(r^1)$. For any j and any z, $\left[F(r^2)\right]^j - \left[F(r^1)\right]^j$ is weakly increasing in $F(r^1)$. To see this, write the difference as

$$\begin{aligned}
\left[F(r^2)\right]^j - \left[F(r^1)\right]^j &= \left[F(r^1) + z\right]^j - \left[F(r^1)\right]^j \\
&= \sum_{k=0}^{j} \binom{j}{k} \left[F(r^1)\right]^{(j-k)} z^k - \left[F(r^1)\right]^j \\
&= \sum_{k=1}^{j} \binom{j}{k} \left[F(r^1)\right]^{(j-k)} z^k,
\end{aligned}$$

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which is increasing in $F(r^1)$ (weakly if $z = 0$).

As $F(r)$ is continuous by assumption and since $p(a, \epsilon; r)$ only depends on r through $F(r)$ it is sufficient to prove that p is continuous in F for all $F \in [0, 1]$. Fix an $\varepsilon > 0$. Assumption 1 implies that there is a J such that $\sum_{j=J}^{\infty} jq\left(j; a, \epsilon\right) < \varepsilon / \left(2\mu\right)$. Therefore, for any $F(r^1) \leq F(r^2) \in [0, 1]$,

$$p(a, \epsilon; r^2) - p(a, \epsilon; r^1) < \mu^{-1} \sum_{j=0}^{J-1} jq\left(j; a, \epsilon\right) \left\{ \left[F(r^2)\right]^{j-1} - \left[F(r^1)\right]^{j-1} \right\} + \varepsilon/2.$$

Following the discussion above, for a given $z = F(r^2) - F(r^1)$, the difference $\left[F(r^2)\right]^{j-1} - \left[F(r^1)\right]^{j-1}$ is largest for $F(r^2) = 1$. Moreover, $1 - \left[F(r^1)\right]^{j}$ is weakly increasing in j. Therefore,

$$p(a, \epsilon; r^2) - p(a, \epsilon; r^1) < \mu^{-1} \left[1 - \left(1 - z\right)^{J-2}\right] \sum_{j=0}^{J-1} jq\left(j; a, \epsilon\right) + \varepsilon/2.$$

As all q's are positive, assumption 1 implies $\sum_{j=0}^{J-1} jq\left(j; a, \epsilon\right)$ is finite. By choosing $z = F(r^2) - F(r^1)$ to be sufficiently small, the first term in the expression above can be made less than $\varepsilon/2$. Hence, p is continuous in F. ∎

The following proof is similar to that of lemma 1 of Burdett and Judd (1983).

Proof of proposition 1. $F(\cdot)$ is continuous. Suppose to the contrary that there is a mass of firms offering the same return r. Then any one of those firms could profitably deviate to $r + \varepsilon$ for some sufficiently small $\varepsilon > 0$. To see this, write the profits as

$$\pi(r) = \left(1 - \frac{1 + r + \varepsilon}{A}\right) \sum_{\epsilon} \int h(a, \epsilon) p(a, \epsilon; r) \Phi(da, \epsilon)$$
$$p(a, \epsilon; r + \varepsilon) = \mu^{-1} \sum_{j=0}^{\infty} jq\left(j; a, \epsilon\right) \left[F(r + \varepsilon)\right]^{j-1}.$$

Note that as F is discontinuous at r, $p(a, \epsilon; r + \varepsilon)$ is discretely larger than $p(a, \epsilon; r)$ at all (a, ϵ), but $\left[1 - \left(1 + r + \varepsilon\right)\left(A\right)\right]$ is only lower than $\left[1 - \left(1 + r\right)\left(A\right)\right]$ by the arbitrarily small amount ε/A so there is a sufficiently small ε for which $r + \varepsilon$ represents a profitable deviation from r.

The support of $F(\cdot)$ starts at zero. An offer below the reservation return would not attract any investments so the expected profit (before the fixed cost) is zero while offering a return of zero (the reservation return) would yield a positive expected profit as, by assumption 2(i), some households only encounter a single firm. Thus, negative returns are dominated so the support of the offer distribution does not begin below zero. Now consider a firm making an offer r for which $F(r) = 0$. Such a firm only receives investments from households who have encountered no other firms so the firm can offer the reservation return and receive no fewer investments.

The support of $F(\cdot)$ ends at some $\bar{r} < A - 1$. Suppose to the contrary that $\bar{r} \geq A - 1$. Firms offering $A - 1$ or more would make zero profits at best and a deviation to offering the reservation return would be profitable.

$F(\cdot)$ is strictly increasing on $[0, \bar{r}]$. Consider two points $r^1 < r^2 \in [0, \bar{r}]$. Suppose to the contrary that $F(r^1) = F(r^2)$. If $F(r^1) = 1$ then $r^1 \geq \bar{r}$ and $r^2 > \bar{r}$. Thus, $F(r^1) < 1$. As $F(\cdot)$ is continuous, for any $\varepsilon > 0$, there must be some $\tilde{r}^2 \geq r^2$ such that $F(\tilde{r}^2) < F(r^1) + \varepsilon$ and $\tilde{r}^2$ is offered in equilibrium. As $p(a, \epsilon; r)$ is continuous in r, for any $\varepsilon' > 0$ there is an ε such that $p(a, \epsilon; \tilde{r}^2) - p(a, \epsilon; r^1) < \varepsilon'$. Therefore, using the notation $S(r) = \pi(r) \times A / (A - 1 - r)$,

$$
\begin{aligned}
S(\tilde{r}^2) - S(r^1) &= \sum_{\epsilon} \int h(a, \epsilon) \left[p(a, \epsilon; \tilde{r}^2) - p(a, \epsilon; r^1)\right] \Phi(da, \epsilon) \\
&< \varepsilon' \sum_{\epsilon} \int h(a, \epsilon) \Phi(da, \epsilon).
\end{aligned}
$$

Assumption 2(iii) implies the integral is a finite constant so this difference can be made arbitrarily small through an appropriate choice of ε' and ε. Now consider the profit from offering $\tilde{r}^2$ rather than r^1.

$$
\begin{aligned}
\pi\left(\tilde{r}^2\right) - \pi\left(r^1\right) &= \left(1 - \frac{1 + \tilde{r}^2}{A}\right) S(\tilde{r}^2) - \left(1 - \frac{1 + r^1}{A}\right) S(r^1) \\
&= \left(1 - \frac{1 + \tilde{r}^2}{A}\right) \left[S(\tilde{r}^2) - S(r^1)\right] + \frac{r^1 - \tilde{r}^2}{A} S(r^1).
\end{aligned}
$$

For sufficiently small ε, the first term in the expression above can be made arbitrarily small while the second is fixed and negative as $S(r^1)$ is positive by assumption 2(i). This implies for small ε, $\pi(r^1) > \pi(\tilde{r}^2)$ so $\tilde{r}^2$ is not offered in equilibrium, but $\tilde{r}^2$ is offered by construction, hence a contradiction. ∎

B Heterogeneous firms

It is assumed above that all firms have marginal product of capital A. The goal of this appendix is to show that the results are robust to firm heterogeneity. To accomplish this, this appendix considers a sequence of firm equilibria in which the firms' marginal products are distributed on the sequence of intervals $\left[\underline{A}_n, \bar{A}_n\right]$. The main result shows that if $\underline{A}_n \to A$ and $\bar{A}_n \to A$ for some A then the associated sequence of offer distributions converges to the one that arises in the case in which firms are homogeneous with marginal product of capital A. In this appendix the behavior of households is taken as given, which implies that there is a unique firm equilibrium. In addition, the behavior of households is assumed to satisfy assumption 2.

Let $\{D_n(A)\}_{n=1}^{\infty}$ be a sequence of distributions of marginal products with associated supports $\left\{\left[\underline{A}_n, \bar{A}_n\right]\right\}_{n=1}^{\infty}$ that satisfy $\underline{A}_n \to A$ and $\bar{A}_n \to A$ as $n \to \infty$. Let $\{F_n(r)\}_{n=1}^{\infty}$ be the sequence of offer distributions associated with the sequence of marginal product distributions $\{D_n(A)\}_{n=1}^{\infty}$. The goal is to show that $\{F_n(r)\}_{n=1}^{\infty} \to F(r)$ point-wise as $n \to \infty$, where $F(r)$ is an offer distribution that arises in the equilibrium with homogeneous firms.

As a matter of notation, note that equation 2 can be rewritten as

$$\pi_n(r, A) = \left(1 - \frac{1+r}{A}\right) S_n(r),$$

where $S_n(r) = \int\int h(a, \epsilon) p_n(a, \epsilon; r) \Phi(da, d\epsilon)$ represents the expected investments when a return r is offered, which is independent of the marginal product of capital. $p_n(a, \epsilon; r)$ is given by

$$p_n(a, \epsilon; r) = \left\{ \mu^{-1} \sum_{j=0}^{\infty} jq\left(j; a, \epsilon\right) \left[F_n(r)\right]^{j-1} \right\}.$$

Notice that $S_n(r)$ depends on the offer distribution $F_n(r)$ through $p_n(a, \epsilon; r)$. $S(r)$ will refer to the same object in the homogeneous case. In addition, let $\bar{r}_n = \sup\{r : F_n(r) < 1\}$ and for the homogeneous case let $\bar{r} = \sup\{r : F(r) < 1\}$.

Before stating and proving this proposition, a few lemmas are necessary.

Lemma 2. *For any n, $F_n(r)$ is continuous and strictly increasing with* $\inf\{r : F(r) > 0\} = 0$.

Proof. The proof is the same as for the homogeneous case. $\blacksquare$

Lemma 3. *$S_n(r)$ is continuous for any n.*

Proof. This follows from lemmas 1 and 2. $\blacksquare$

The fact that $F_n(r)$ is strictly increasing does not necessarily imply that all points in the interval $[0, \bar{r}_n]$ are offered in equilibrium as a single point of measure zero could be omitted from the support of $F_n(r)$ without difficulty. This possibility motivates the next lemma, which states that the support of $F_n(r)$ is dense in $[0, \bar{r}_n]$.

Lemma 4. *For any n and any $\varepsilon > 0$, if there are three points x, $\underline{x}$ and $\bar{x}$ such that $\underline{x}$ and $\bar{x}$ are in the support of $F_n(r)$ and $\underline{x} \leq x \leq \bar{x}$, then there is a point y in the support of $F_n(r)$ such that $|x - y| < \varepsilon$.*

Proof. If $|x - \bar{x}| < \varepsilon$ or $|x - \underline{x}| < \varepsilon$ the result is immediate. Otherwise, suppose to the contrary that there were points x, $\underline{x}$ and $\bar{x}$ such that $\underline{x}$ and $\bar{x}$ are in the support of $F_n(r)$ and $\underline{x} \leq x \leq \bar{x}$, but there is no such point y in the interval $(x - \varepsilon, x + \varepsilon)$. Let $\bar{x}' = \sup\{r : F_n(r) = F_n(x)\}$ and $\underline{x}' = \inf\{r : F_n(r) = F_n(x)\}$. By assumption $\bar{x}' - \underline{x}' > 2\varepsilon$. By construction there are points in the support of F_n that are arbitrarily close to $\bar{x}'$ and $\underline{x}'$. As $F_n(\bar{x}') = F_n(\underline{x}')$ it follows that profits from offering $\underline{x}'$ are strictly greater than those from offering $\bar{x}'$ for any marginal product A, which, by the continuity of $F_n(r)$ and $\pi(r, A)$, is inconsistent with points in a neighborhood around $\bar{x}'$ being in the support of $F_n(r)$. $\blacksquare$

Lemma 5. *For any n and x, if there are points $\bar{x} \geq x$ and $\underline{x} \leq x$ such that $\bar{x}$ and $\underline{x}$ are in the support of*

F_n then $S_n(x)$ must satisfy the following inequalities

$$\frac{\bar{A}_n - 1}{\bar{A}_n - 1 - x} \leq \frac{S_n(x)}{S_n(0)} \tag{1}$$
$$\frac{\underline{A}_n - 1}{\underline{A}_n - 1 - x} \geq \frac{S_n(x)}{S_n(0)}. \tag{2}$$

Proof. There are two cases to consider: either x is in the support of F_n or $\underline{x} < x < \bar{x}$ and x is not in the support of F_n. Suppose x is in the support of F_n. Let A^1 denote the marginal product of the firm that offers x. This firm must not wish to deviate to offer a return of 0. That is

$$\left(A^1 - 1\right) S_n(0) \leq \left(A^1 - 1 - x\right) S_n(x),$$

which can be rearranged as

$$\frac{A^1 - 1}{A^1 - 1 - x} \leq \frac{S_n(x)}{S_n(0)}.$$

And $A^1 \leq \bar{A}_n$ implies inequality (1). Analogous steps lead to inequality (2).

Now consider the case in which x is not in the support of F_n and $\underline{x} < x < \bar{x}$. By using lemma 4 repeatedly, it is possible to construct a sequence $\{x_i\}_{i=1}^{\infty}$ that converges to x with the property that every element of the sequence is in the support of F_n. The argument above applies to all such points. By the Comparison Theorem for Functions,[1] it follows that $\lim_{x_i \to x} S_n(x_i)$ satisfies the inequalities. As $S_n(\cdot)$ is continuous, $S_n(x) = \lim_{x_i \to x} S_n(x_i)$. ∎

Lemma 6. *$\bar{r}_n \to \bar{r}$ as $n \to \infty$.*

Proof. In the homogeneous case, firms are indifferent between offering all returns and in particular the

[1]See Wade (2000), p. 62.

highest and lowest returns so $\bar{r}$ must satisfy the condition

$$(A-1)\,S(0)=(A-1-\bar{r})\,S(\bar{r}). \tag{3}$$

As $F(\bar{r})=1$, $S(\bar{r})$ is given by

$$S(\bar{r})=\mu^{-1}\int\int h(a,\epsilon)\sum_{j=0}^{\infty} jq\,(j;a,\epsilon)\,\Phi(da,d\epsilon), \tag{4}$$

which is independent of the offer distribution. Similarly, as $F(0)=0$, $S(0)$ is also independent of the offer distribution and is given by

$$S(0)=\mu^{-1}\int\int h(a,\epsilon)q\,(1;a,\epsilon)\,\Phi(da,d\epsilon). \tag{5}$$

Finally, equation (3) can rearranged as

$$\bar{r}=A-1-(A-1)\,\frac{S(0)}{S(\bar{r})}, \tag{6}$$

where $S(0)$ and $S(\bar{r})$ are given by the expressions above.

The proof now proceeds by showing that for any $\varepsilon>0$ there is an N such that $n>N$ implies the support of F_n lies below $\bar{r}+\varepsilon$ and there is a point x in the support of F_n that satisfies $|x-\bar{r}|<\varepsilon$. Consider a firm that offers $\bar{r}+\varepsilon$ or more. Such a firm must not have an incentive to deviate to offering a return of zero. That is

$$\left(A^1-1-\bar{r}-\varepsilon\right)S_n(\bar{r}+\varepsilon)\geq\left(A^1-1\right)S_n(0)$$

for some $A^1\in\left[\underline{A}_n,\bar{A}_n\right]$. As $F_n(\bar{r}+\varepsilon)\leq 1$, it must be the case that $S_n(\bar{r}+\varepsilon)\leq S(\bar{r})$ as given in equation (4). Moreover, it is still the case that $F_n(0)=0$ so $S_n(0)=S(0)$ as given in equation (5). Making these substitutions, substituting equation (6) for $\bar{r}$ and rearranging yields

$$\left(A^1 - A\right)\left(1 - \frac{S(0)}{S(\bar{r})}\right) \geq \varepsilon.$$

The first term on the left-hand side goes to zero as $n \to \infty$ and the second term is a positive constant. Therefore, this condition cannot hold for any $\varepsilon > 0$ as $n \to \infty$. Thus, $\lim_{n\to\infty} \bar{r}_n \leq \bar{r}$.

Suppose that the support of F_n lies entirely below $\bar{r} - \varepsilon$ for some $\varepsilon > 0$. Then $S_n(\bar{r} - \varepsilon) = S(\bar{r})$. Moreover, it is still the case that $S_n(0) = S(0)$. Now consider a firm that offers a return of arbitrarily close to zero (by lemma 2 there must be such a firm). This firm must not have an incentive to deviate to offering $\bar{r}$, which any firm is free to do. That is

$$\left(A^1 - 1\right) S(0) \geq \left(A^1 - 1 - \bar{r} + \varepsilon\right) S(\bar{r})$$

inserting equation (6) for $\bar{r}$ and rearranging yields

$$\left(A^1 - A\right)\left(\frac{S(0)}{S(\bar{r})} - 1\right) \geq \varepsilon.$$

The left-hand side of this expression goes to zero as $n \to \infty$ so the condition does not hold in the limit. Thus, $\lim_{n\to\infty} \bar{r}_n \geq \bar{r}$ and so $\lim_{n\to\infty} \bar{r}_n = \bar{r}$. ∎

Proposition 1. *Let* $\{D_n(A)\}_{n=1}^{\infty}$ *be a sequence of distributions of marginal products with associated supports in* $\left\{\left[\underline{A}_n, \bar{A}_n\right]\right\}_{n=1}^{\infty}$*. For each* $D_n(A)$*, there is an offer distribution* $F_n(r)$*. As* $\underline{A}_n \to A$ *and* $\bar{A}_n \to A$*,* $F_n(r) \to F(r)$*, where* $F(r)$ *is the offer distribution that arises when all firms are homogenous with marginal product* A*.*

Proof. Fix a point $x \in (0, \bar{r})$. By lemma 6 there is an N such that for $n > N$ there are points $\underline{x}_n$ and $\bar{x}_n$ in the support of F_n such that $\underline{x}_n \leq x \leq \bar{x}_n$. Therefore lemma 5 applies and for $n > N$ it is the case that

$$\frac{\bar{A}_n - 1}{\bar{A}_n - 1 - x} \leq \frac{S_n(x)}{S_n(0)} \leq \frac{\underline{A}_n - 1}{\underline{A}_n - 1 - x}$$

Recall that $S_n(0) = S(0)$ for all n, so by the Squeeze Theorem it follows that

$$\lim_{n\to\infty} S_n(x) = \frac{A-1}{A-1-x} S(0). \tag{7}$$

From the equal-profit condition in the homogeneous case we have

$$(A - 1 - x)\, S(x) = (A - 1)\, S(0),$$

so equation (7) implies $S_n(x) \to S(x)$.

Notice that $S_n(\cdot)$ depends on n only through F_n so it may be written as $S\left(F_n\left(\cdot\right)\right)$. Given assumption 2, $S(\cdot)$ is continuous and strictly increasing in its argument $F \in [0, 1]$ and is therefore invertible. As a result, convergence of $S_n(x)$ to $S(x)$ implies convergence of $F_n(x)$ to $F(x)$. ∎

C Computational details

C.1 Computing the steady state

For a given set of parameter values, the algorithm begins with a guess of the capital-labor ratio and the associated wage and marginal product of capital. Next I guess a distribution of offered returns. The consumer's problem can then be solved and simulated. After solving the household's problem, the next step is to simulate a sample of households from which one can compute the corresponding firm equilibrium offer distribution from the firm's profit equation. I then solve the household problem again until the offer distribution converges (it usually does so after a small number of iterations). Finally, I check the simulated capital-labor ratio and updates the guess.

To solve the household's problem I use a value function iteration algorithm in which the value function for each discrete type is interpolated over 35 unequally spaced nodes using cubic splines. Solving the household's problem requires integrating over returns with respect to $G(r, s)$. To compute these integrals I use Chebyshev quadrature with 21 nodes over $(0, \bar{r})$ and an additional node at $r = 0$. To compute the density of $G(r, s)$ at each node, I differentiate equation (5), which gives the density as a function of s, $F(r)$ and the density of $F(r)$. The latter two objects are stored when I solve the firm's problem.

Simulating the household's problem requires sampling from the distribution $G(r, s)$. I do this by approximating $F(r)$ with a discrete grid of points from which equation (5) gives a discrete approximation to $G(r, s)$ for a given level of search effort. I then draw from a uniform distribution and numerically invert $G(r, s)$ using this discrete approximation.

Computing the firm equilibrium simply requires solving a differential equation. As firms are indifferent between offered returns, the derivative of the firm's profit equation with respect to r must be zero on the support of $F(r)$. Rearranging this derivative produces a differential equation that $F(r)$ must satisfy. Proposition 1 provides an initial condition of $F(0) = 0$ that can be used to solve for $F(r)$.

C.2 Computing the transition

The computational procedure used to find the transition path again resembles the one would use to compute the transition of a standard Bewley-Huggett-Aiyagari economy. First, one chooses a number of periods after which the economy is assumed to be in the new steady state. Next one guesses on a path for the capital-labor ratio. Given the capital-labor ratio, one guesses a sequence of offer distributions. One then solves the consumer problem backwards from the new steady state to find the households' decision rules and then uses those decision rules to simulate forward from the initial distribution of households over the state space. Using the simulated sample, one can compute solve the firm's problem for a new sequence of offer distributions. The algorithm updates the offer distributions and iterates until they converge while holding the capital-labor ratio fixed. Once this has been accomplished one compares the simulated capital-labor ratio to the guess and updates the guess. This procedure is repeated until the simulated capital labor ratio

age group	$21-25$	$26-30$	$31-35$	$36-40$	$41-45$	$46-50$	$51-55$	$56-60$	$61-65$
$\bar{y}_i$	1.00	1.28	1.45	1.56	1.62	1.63	1.59	1.53	1.38

Table 1: Life-cycle profile in labor productivity.

matches the guess. An outer loop checks that the sequence of consumption taxes clears the social security budget and updates the tax sequence if it does not. Finally, one must check that the economy does in fact reach the new steady state in the number of periods assumed at the outset.

D Estimating the life-cycle profile of labor productivity

To estimate the deterministic life-cycle profile in labor productivity I use data from the 1968 - 2005 waves of the PSID. I create a measure of the household wage by summing the total labor income of the head and wife and dividing by the total hours of the head and wife. I require that households work at least 208 hours to be included in the sample and that the calculated wage be at least one-half of the minimum wage in the year in question. Households are divided into nine age groups 21-25, ..., 61-66 and I regress the log wage on year and age group dummies. The resulting life-cycle profile for wages appears in Table 1.

References

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